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Symplectic coherent-state bases in the OCM calculations of nuclear cluster systems

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In order to solve many-body states in a wide energy region including continuum states, it is strongly desired to develop a useful method. In nuclear cluster systems [1, 2,], we must treat both bound and unbound states simultaneously. For such calculations, we propose to apply symplectic coherent-state bases (SCSB). In this report, the basic idea of the SCSB method will be explained. Especially the Pauli exclusion principle plays an important role in nuclear systems [3, 4], and the orthogonality condition model (OSM) [5] has been studied successfully for various kinds of nuclear cluster systems. We here discuss an application of the SCSB to the OCM. The SCSB method can be easily applied to two-cluster systems. However, for multi-cluster systems, the applicability of the SCSB is not self-evident. The multi-cluster system is described by the Pauli-allowed states (PAS) [6] under the Pauli exclusion principle. In this report, the multi-cluster SCSB will be discussed on the basis of the PAS of the multi-cluster systems.

Key words: cluster model, orthogonality condition model, symplectic-coherent basis state.
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1 Introduction

To obtain information on bound and resonance states in two-cluster and multi-cluster nuclear structures, it is necessary to calculate the Schrödinger equation. In addition, a prerequisite for obtaining a result that is closest to reality is taking into account the Pauli principle, which leads to a complication of the solution for the total wave function of the nuclear system. This equation, like the Schrödinger equation, is solved within the orthogonality condition model. However, using the method of symplectic coherent-state bases, could simplify the process of calculating this problem. The method of symplectic coherent-state bases has a number of features that, within the framework of the orthogonality condition model, would allow simultaneous calculations for both bound and resonant states. Another interesting question is the applicability of this method to solving multi-cluster nuclear systems. The multi-cluster system is described by the Pauli-allowed states (PAS) for the relative motion of clusters under the Pauli exclusion principle for nucleons constituting clusters [7, 8, 9, 10]. We discuss the multi-cluster SCSB on the basis of the PAS of the multicluster systems [11].

In the next section, we will briefly explain the coherent states [12, 13, 14], and propose the symplectic coherent-state bases. In Sec. 3, the applications of the symplectic coherent-state bases are discussed for two-body and many-body cluster systems [15]. Finally, summary is given in Sec. 4.

2 Symplectic coherent-state bases (SCSB)

2.1 What is a coherent state

The coherent state was first proposed by Schrodinger in 1926 [16], and has been widely used in quantum physics so far. The characteristic of the coherent state is in description of a classical property of the quantum oscillation.

Let's start with a one-dimensional Harmonic Oscillator (HO) system, which is described by the Hamiltonian

$$H = \frac{1}{2}\left(p_x^2 + x^2\right) = \frac{1}{2}\left(a^\dagger a + a a^\dagger\right), \quad (1)$$

where

$$a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right), a = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right). \quad (2)$$

These creation and anniheration operators satisfy the following canonical commutation relation (CCR):

$$\left[a, a^\dagger\right] = 1, [a, 1] = 0, \left[a^\dagger, 1\right] = 0, \quad (3)$$

Here, we introduce three types of definitions of the coherent state.

(I) Geometrical definition

For a state (x), the state displaced by c from x is expressed as

$$\psi(x - c) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!}\frac{\partial^n}{\partial x^n}\psi(x), = \exp\left(-c\frac{\partial}{\partial x}\right)\psi(x),\ e^{\frac{c}{\sqrt{2}}(a^\dagger - a)}\psi(x). \quad (4)$$

Here, $c/\sqrt{2}$ is replaced by a new complex valuable z, and we present a generalized displacement operator as

$$D = z^* a^\dagger - za. \quad (5)$$

Thus, we have a definition of the coherent state

$$|z\rangle_I = e^D|0\rangle, \quad (6)$$

where $|0\rangle$ is a vacuum state defined by $a|0\rangle = 0$ for the HO operator a.

(II) Generating function

In the second definition of the coherent state using the generating function, we have

$$|z\rangle_{II} = e^{z^* a^\dagger}|0\rangle = \sum_{n=0}^{\infty}\frac{(z^*)^n}{\sqrt{n!}}|n\rangle, \quad (7)$$

Where $|n\rangle$ is a HO state of $|n\rangle = \frac{1}{\sqrt{n}}a^\dagger|n-1\rangle$. The overlap is given as

$$\langle z'|z\rangle = e^{z' z^*}, \quad (8)$$

and, in general, the coherent states are overcomplete. The unitary operator is also given by

$$1 = \frac{1}{\pi}\int d^2 z e^{-zz^*}|z\rangle\langle z|. \quad (9)$$

A state vector $|\psi\rangle$ can be expressed in the z-space function realization;

$$\langle z|\psi\rangle = \psi(z). \quad (10)$$

This z-space is called as the Bargmann space [17], and

$$\psi(z) = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}c^n, \quad (11)$$

where $c_n = \langle n|\psi\rangle$. The relation between the functions in real (x) and Bargmann (z) spaces is given by the Bargmann transformation

$$\psi(z) = \int dx A(x,z)\psi(x), \quad (12)$$

where

$$A(x,z) = \frac{1}{\pi^{1/4}} exp\left(-\frac{1}{2}x^2 + \sqrt{2x}z - \frac{1}{2}z^2\right). \quad (13)$$

(III) An eigenstate of the operator a an eigenstate of the non-hermitian operator a ($a^\dagger \neq a$) has the eigenvalue of a complex number:

$$a|z\rangle_{III} = z^*|z\rangle_{III}. \quad (14)$$

The equivalence between this and the second definitions is shown as

$$a|z\rangle_{II} = a\sum_{n=0}^{\infty}\frac{(z^*)^n}{\sqrt{n!}}|n\rangle = \sum_{n=1}^{\infty}\frac{(z^*)^n}{\sqrt{(n-1)!}}|n-1\rangle =$$

$$= z^*\sum_{n=1}^{\infty}\frac{(z^*)^{(n-1)}}{\sqrt{(n-1)!}}|n-1\rangle = z^*\sum_{n=0}^{\infty}\frac{(z^*)^n}{\sqrt{n!}}|n\rangle = z^*|z\rangle_{II} \quad (15)$$

2.2 Generalised coherent state

The generalised coherent state is proposed by Perelomv [18]. For an arbitrary Lie group G, we express its unitary irreducible representation of $g = G$ in the Hilbert space H as $U(g)$. The generalised coherent states $|\psi_g\rangle$ is defined by

$$|\psi_g\rangle = U(g)|\psi_0\rangle, \qquad (16)$$

where $|\psi_0\rangle$ is a fixed vector in the Hilbert space H.

2.3 Symplectic coherent state

Let us consider the three-dimensional HO system, in which creation and annihilation operators are expressed by $\boldsymbol{a}^\dagger$ and $\boldsymbol{a}$. Using the operator

$$D^\dagger = \boldsymbol{a}^\dagger\boldsymbol{a}^\dagger, \qquad (17)$$

we introduce a coherent state based on the above type II definition as

$$\exp(\beta D^\dagger)|N\ell\nu\rangle = \sum_{k=0}^{\infty}\frac{1}{\sqrt{k!}}\beta^k\left(D^\dagger\right)^k|N\ell\nu\rangle, \qquad (18)$$

where $|N\ell\nu\rangle$ is the three-dimensional HO wave function with the total oscillator quanta N, orbital angular momenta ℓ and the oscillator constant ν.

The three-dimensional HO wave function [19] can be expressed by using creation operators as

$$|N\ell\nu\rangle = A_{n\ell}\left(\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger\right)^n\mathcal{Y}_{\ell m}(\boldsymbol{a}^\dagger)|0\rangle, \qquad (19)$$

where

$$A_{n\ell} = (-1)^n\sqrt{\frac{4\pi}{(2n+2l+1)!!\,(2n)!!}}, \qquad (20)$$

$$\mathcal{Y}_{\ell m}(x) = x^\ell Y_{\ell m}(\hat{x})|0\rangle, \qquad (21)$$

and $N = 2n + \ell$. Thus, the coordinate representation of the coherent state can be written as

$$|\beta; N\ell m\rangle \equiv \exp\left(\frac{\beta}{2}D^\dagger\right)|N\ell\nu\rangle = \left(\frac{1}{\sqrt{1+\beta}}\right)^{N+\frac{3}{2}}\exp\left(\frac{\nu\beta}{1+\beta}r^2\right)\left|N\ell\frac{\nu}{1+\beta}\right\rangle. \qquad (22)$$

3 Application for cluster models

In a two-cluster system, the relative motion between clusters is described by the Hamiltonian

$$H = T + V(r), \qquad (23)$$

and the Schrodinger equation is given as

$$H\Phi_\alpha = E_\alpha\Phi_\alpha. \qquad (24)$$

Using the symplectic coherent states $|\beta; N\ell m\rangle$ as basis functions;

$$\Phi_\alpha = \sum_{i=1}^{imax} c_i^\alpha\phi_{i\ell}^{N_0\nu_0}(r), \qquad (25)$$

where

$$\phi_{i\ell}^{N_0\nu_0}(r) = |\beta_i; N_0\ell\nu_0\rangle, \qquad (26)$$

we can solve the eigenvalue problem of the Schrodinger equation.

When we chose $N_0 = \ell$ for the fixed state $|N_0\ell\nu_0\rangle$, the symplectic coherent states are written as

$$\phi_{i\ell}^{N_0\nu_0}(r) = \left(\sqrt{1-\beta_i^2}\right)^{\ell+\frac{3}{2}} \times \exp\left(\frac{\beta_i}{2}D^\dagger\right)|(N_0 == \ell)\ell\nu_0\rangle. \qquad (27)$$

Here, we put

$$\beta_i = \frac{\left(\gamma^{i-1}\right)^2 - 1}{(\gamma^{i-1})^2 + 1}, \nu_0 = \frac{1}{b_0^2}, \qquad (28)$$

then the symplectic coherent states are

$$\phi_{i\ell}^{N_0\nu_0}(r) = \phi_{i\ell}^G(r)Y_{\ell m}(\hat{r}), \qquad (29)$$

where

$$\phi_{i\ell}^G(r) = N_{i\ell}r^\ell e^{-\nu_i r^2}, \nu_i = \left(\frac{1}{b_0\gamma^{i-1}}\right)^2. \qquad (30)$$

This is the Gaussian basis states which have widely been used as basis functions to solve the Schrodinger equation.

When we chose $|N_0\ell\nu_0\rangle$ for the fixed state, the symplectic coherent states $|\beta_i; N_0\ell\nu_0\rangle$ satisfy

$$\langle N\ell\nu_0|\beta_i; N_0\ell\nu_0\rangle = 0 \text{ for } N < N_0, \quad (31)$$

because $|\beta_i; N_0\ell\nu_0\rangle$ includes the HO states of $N' \geq N_0$. This means that the symplectic coherent states $|\beta_i; N_0\ell\nu_0\rangle$ describe the Pauli allowed states orthogonal to the Pauli forbidden states $|N_F\ell\nu_0\rangle$ where $N_F < N_0$.

For the multi-cluster system, the antisymmetrized wave function is given by

$$\Psi = \boldsymbol{A}\{\varphi_1\varphi_2 \dots \varphi_n \Phi(r_1, r_2, \dots, r_{n-1})\}, \quad (32)$$

Where φ_k (k = 1, … , n) and $\Phi(r_1, r_2, \dots, r_{n-1})$ are internal and relative wave functions of clusters, respectively. The orthogonality condition model (OCM) is also given as

$$H_{OCM}\Phi = E\Phi, \text{ with } \langle \Phi|\Phi_F\rangle = 0, \quad (33)$$

where ϕ_F are the Pauli forbidden states defined by

$$\boldsymbol{A}\{\varphi_1\varphi_2 \dots \varphi_n \Phi_F\} = 0. \quad (34)$$

The Pauli allowed states Φ_A are defined by $\langle \Phi_A|\Phi_F\rangle = 0$as well.

The Pauli-allowed symplectic coherent states can be constructed by using the operator

$$D^\dagger = \sum_{k=1}^{n} \boldsymbol{a}_k^\dagger \boldsymbol{a}_k^\dagger - \boldsymbol{a}_G^\dagger \boldsymbol{a}_G^\dagger, \quad (35)$$

where the last term is due to degree of freedom of the center of mass. The fixed states are chosen to be the lowest Pauli allowed states, which have been discussed in Ref. [2].

The symplectic coherent states are closely connected with the complex scaling method in which continuum states can be described satisfactorily within the square-integrable functions [20, 21]. The complex scaling [22] is defined by

$$r \to re^{i\theta}, \Phi(\mathrm{r}) \to e^{3i\theta/2}\Phi(\mathrm{r}e^{i\theta}). \quad (36)$$

This transformation is a kind of dilation, which is

$$\Phi(\vec{r}) = \exp(-\alpha\vec{r}\cdot\vec{\partial})\Phi(\vec{r}), \quad (37)$$

where

$$\vec{r}\cdot\vec{\partial} = -\frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger + \frac{1}{2}\boldsymbol{a}\boldsymbol{a} - \frac{3}{2}N. \quad (38)$$

Therefore, for the lowest Pauli allowed state $|N_0\ell\nu_0\rangle$, we have

$$\begin{aligned} &\exp(-\alpha\vec{r}\cdot\vec{\partial})\,|N_0\ell\nu_0\rangle = \\ &= \exp\left[\frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}^\dagger + \frac{3}{2}N\right]|N_0\ell\nu_0\rangle = \\ &= \exp\left(\frac{3}{2}\alpha N_0\right)|\alpha; N_0\ell\nu_0\rangle \end{aligned} \quad (39)$$

Thus we can see that the complex scaling is also a symplectic coherent state of dilation with a complex number parameter.

4 Conclusions

Many states observed around the threshold energies in light nuclei have been studied to have characteristic cluster structures [23]. In this report, a promising method using the symplectic coherent-state bases (SCSB) has been discussed to describe those states, which are weakly bound and resonant states. The SCSB method can be easily applied to two-cluster systems. However, for multicluster systems, the applicability of the SCSB is not self-evident. The multi cluster system is described by the Pauli-allowed states (PAS) [6] for the relative motion of clusters under the Pauli exclusion principle for nucleons constituting clusters. We explained the symplectic coherent sates based on the lowest Pauli allowed states in twobody system. Using the symplectic coherent sates as basis states, we proposed to solve the OCM equation. For this, the Schrödinger equation describing the states and relative motion of two cluster systems was considered in detail. We also discussed that this idea can be extended to the multi-cluster OCM problem.

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