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Non-rotating and slowly rotating white dwarfs in classical physics

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The equilibrium configurations of uniformly rotating white dwarfs are calculated in the framework of classical physics. The Chandrasekhar and the Salpeter equations of state are used to describe the white dwarf matter. The Hartle formalism is applied to the integration of the equations of hydrostatic equilibrium and field equations. The equations of structure have been expanded in powers of the angular velocity Ω of the white dwarf, and terms of higher order than Ω^2 have been neglected. All parameters of rotating white dwarfs such as the total mass, polar and equatorial radii, eccentricity, moment of inertia, angular velocity, angular momentum, gravitational potential and quadrupole moment have been calculated numerically within this approximation.

Key words: white dwarfs, uniform rotation, Hartle's formalism, equilibrium configurations, moment of inertia, quadrupole moment.
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1 Introduction

In order to investigate the equilibrium structure of non-rotating and rotating white dwarfs in the Newtonian physics a number of authors have calculated in greater detail the classical equilibrium configurations of cold white dwarfs [1, 2]. In the literature among all those approaches the Hartle formalism in the classical case has been neglected, probably due to its relativistic counterpart, which is widely used in the scientific community to describe relativistic objects such as neutron stars, quark stars, and other exotic objects [3-5].

In our recent work [6] we have revisited the Hartle formalism in the classical case, giving detailed derivations of all the physical quantities such as the total rotating mass, the equatorial and polar radii, eccentricity, moment of inertia both for non-rotating and rotating configurations, and quadrupole moment. All these parameters play a pivotal rolein the investigation of the stability and the lifespan of white dwarfs, main sequence stars and giant stars[7-9].

In this work we investigate the effects of angular velocity on the structure of white dwarfs. We examine the case of white dwarfs which rotate rigidly and slowly. We integrate the equations of structure for slowly rotating white dwarfs numerically for the Chandrasekhar and Salpeter equations of state [10, 11].

2 Equation of structure

The method used to construct models for uniformly and slowly rotating stars is summarized briefly here [3, 6].

Equation of state.
As the first step in the calculation of a slowly rotating stellar model, one-parameter equation of state, $p = p(\rho)$, is specified. Here p is the pressure, ρ is the matter density. For white dwarfs this relation will be one of the equations of state summarized in [11-14].

Non rotating white dwarfs.
For a given value of the central density, the non-rotating equilibrium configuration is determined by integrating the Newtonian equation of hydrostatic equilibrium for the pressure, $p^{(0)}(R)$, and the mass interior to a given radius, $M^{(0)}(R)$:

$$\begin{cases} \frac{dp^{(0)}(R)}{dR} = -\rho(R)\frac{GM^{(0)}(R)}{R^2}, \\ \frac{dM^{(0)}(R)}{dR} = 4\pi R^2 \rho(R). \end{cases} \quad (1)$$

The integration is performed outward, starting at the star's center, $R = 0$. At the star's center $M^{(0)}(R = 0) = 0$; $\rho(R = 0) = \rho_c$ is the given central density; and $p^{(0)}$ is $p^{(0)}(\rho_c)$ as given by the equation of state. The radius of the spherical surface of the star, a, is that value of R at which $p^{(0)}(R)$ drops to zero; and the value of $M^{(0)}(R)$ there is the star's total static mass.

The gravitational potential of a non-rotating star is determined by integrating outward from the center the equation

$$\frac{d\Phi^{(0)}(R)}{dR} = \frac{GM^{(0)}(R)}{R^2} = -\frac{1}{\rho(R)}\frac{dp^{(0)}(R)}{dR} \qquad (2)$$

with the boundary condition $\Phi^{(0)}(\infty) = 0$.

The moment of inertia of a spherical star is calculated easily from the following expression:

$$I^{(0)}(a) = \frac{8\pi}{3}\int_0^a \rho(R)R^4 dR. \qquad (3)$$

Values for the central density and angular velocity.

Once the equation of state is specified, there is a unique equilibrium configuration for each choice of the central density and angular velocity. The small perturbations away from a non-rotating equilibrium configuration are all proportional to the angular velocity or to its square. Consequently, for a given central density, all the models of different angular velocities can be obtained from a single model by applying an appropriate scaling. In this paper the results are given in graphical form for the angular velocity Ω satisfying

$$\Omega = \sqrt{\frac{GM_{tot}}{R_e^3}}, \qquad (4)$$

where G is the gravitational constant, M_{tot} is the total mass of the rotating configuration and R_e is its equatorial radius. This is the critical angular velocity at which mass-shedding will occur, and it is thus a natural upper bound on those angular velocities for which the assumption of slow rotation could be valid. Knowing the values of moment of

inertia $I^{(0)}$ and the angular velocity Ω one can determine the angular moment of a spherical star by

$$J = I^{(0)}(a)\Omega. \qquad (5)$$

Having chosen a value of the angular velocity for each value of the central density, one constructs a sequence of equilibrium models by integrating the Newtonian equations of structure for a sequence of central densities.

The spherical deformation of the star.

The spherical part of the rotational deformation is calculated by integrating the $l = 0$ equations of hydrostatic equilibrium for the "change in mass" $M^{(2)}(R)$ and the "pressure perturbation function" $p_0^*(R)$:

$$\begin{cases} \frac{dp_0^*(R)}{dR} = \frac{2}{3}\Omega^2 R - \frac{GM^{(2)}(R)}{R^2} \\ \frac{dM^{(2)}(R)}{dR} = 4\pi R^2 \rho(R)\frac{d\rho(R)}{dR}p_0^*(R), \end{cases} \qquad (6)$$

These equations are integrated out from the origin with boundary conditions that as $R \to 0$

$$p_0^*(R) \to \frac{1}{3}\Omega^2 R^2, \quad M^{(2)}(R) \to 0. \qquad (7)$$

These boundary conditions guarantee that the central densities of the rotating and non-rotating configurations are the same. Consequently, the total mass of the star with central density ρ_c and angular velocity Ω is

$$M_{tot} = M^{(0)}(a) + M^{(2)}(a), \qquad (8)$$

where a is the radius of the spherical configuration.

The quadrupole deformation of the star.

One calculates the quadrupole part of the deformations by integrating the $l = 2$ equations. Firstly, one needs to find particular solution by integrating equations

$$\begin{cases} \frac{d\chi(R)}{dR} = -\frac{2GM(R)}{R^2}\varphi(R) + \frac{8\pi}{3}\Omega^2 R^3 G\rho(R) \\ \frac{d\varphi(R)}{dR} = \left(\frac{4\pi R^2 \rho(R)}{M(R)} - \frac{2}{R}\right)\varphi(R) - \frac{2\chi(R)}{GM(R)} + \frac{4\pi}{3M(R)}\rho\Omega^2 R^4 \end{cases} \qquad (9)$$

outward from the center with arbitrary initial conditions satisfying equations

$$\varphi(R) \to AR^2,$$
$$\chi(R) \to BR^4,$$
$$B + \frac{2\pi}{3} G\rho_c A = \frac{2\pi}{3} G\rho_c \Omega^2, \qquad (10)$$

where A and B are arbitrary constants. Set, for example, $A = 1$ and define B from the above algebraic equation. This determines particular solutions $\varphi_p(R)$ and $\chi_p(R)$.

Secondly, the homogeneous solution should be considered by integrating the homogeneous equations

$$\begin{cases} \frac{d\chi_h(R)}{dR} = -\frac{2GM(R)}{R^2} \varphi_h(R) \\ \frac{d\varphi_h(R)}{dR} = \left(\frac{4\pi R^2 \rho(R)}{M(R)} - \frac{2}{R}\right) \varphi_h(R) - \frac{2\chi_h(R)}{GM(R)} \end{cases} \qquad (11)$$

outward from the center with arbitrary initial conditions satisfying the equations

$$\varphi_h(R) \to AR^2,$$
$$\chi_h(R) \to BR^4,$$
$$B + \frac{2\pi}{3} G\rho_c A = 0 \qquad (12)$$

Set $A = 1$ and B is given by the above equation. This determines particular solutions $\varphi_h(R)$ and $\chi_h(R)$. Thus interior solution is determined by the sum of the particular and the homogeneous solution

$$\varphi_{in}(R) = \varphi_p(R) + K_2 \varphi_h(R),$$
$$\chi_{in}(R) = \chi_p(R) + K_2 \chi_h(R). \qquad (13)$$

Matching with the Exterior Solutions.
The exterior solutions are given by

$$\varphi_{ex}(R) = \frac{K_1}{R^3}, \quad \chi_{ex}(R) = \frac{K_1 GM^{(0)}}{2R^4}. \qquad (14)$$

By matching (13) and (14) at $(R = a)$

$$\varphi_{ex}(R = a) = \varphi_{in}(R = a),$$
$$\chi_{ex}(R = a) = \chi_{in}(R = a). \qquad (15)$$

constants K_1 and K_2 are determined.

The polar and equatorial radii and eccentricity.
The surface of the rotating configuration, polar R_p and equatorial R_e radii are given by

$$r(a, \Theta) = a + \xi_0(a) + \xi_2(a)P_2(\Theta), \qquad (16)$$

$$R_p = r(a, 0) = a + \xi_0(a) + \xi_2(a), \qquad (17)$$
$$R_e = r(a, \pi/2) = a + \xi_0(a) - \xi_2(a)/2, \qquad (18)$$

where $\xi_0(a)$ and $\xi_2(a)$ are given by

$$\xi_0(R) = \frac{R^2}{GM^{(0)}(R)} p_0^*(R), \qquad (19)$$

$$\xi_2(R) = -\frac{R^2}{GM^{(0)}(R)} \left\{ \frac{1}{3} \Omega^2 R^2 + \varphi_{in}(R) \right\}. \quad (20)$$

The eccentricity is defined by

$$\text{eccentricity} = \sqrt{1 - \left(\frac{R_p}{R_e}\right)^2}. \qquad (21)$$

Quadrupole moment.
The Newtonian potential $\Phi(R, \Theta)$ outside the star $(R > a)$ is given by

$$\Phi(R, \Theta) = -\frac{GM_{tot}}{R} + \frac{K_1}{R^3} P_2(\cos \Theta), \qquad (22)$$

thus the constant K_1 can be written as $K_1 = GQ$, where Q is the mass quadrupole moment of the star. According to Hartle's definition $Q > 0$ defines an oblate object, $Q < 0$ defines a prolate object.

Total moment of inertia and total angular momentum.
The total moment of inertia of a rotating configuration is determined as the sum of the moment of inertia of a static star and the change in the moment of inertia due to rotation and deformation

$$I_{tot}(R) = I^{(0)}(R) + I^{(2)}(R), \qquad (23)$$

where the moment of inertia of the non-rotating star is determined as earlier

$$I^{(0)}(R) = \frac{8\pi}{3} \int_0^R \rho(R) R^4 dR, \qquad (24)$$

and its change due to rotation is given by [15]

$$I^{(2)}(R) = \frac{8\pi}{3} \int_0^R \rho(R) R^4 \left(\frac{d\xi_0(R)}{dR} - \frac{1}{5} \frac{d\xi_2(R)}{dR} + \frac{4}{R} \left[\xi_0(R) - \frac{1}{5} \xi_2(R) \right] \right) dR$$

$$= \frac{8\pi}{3} \int_0^R \left(\left[\frac{1}{5} \xi_2(R) - \xi_0(R) \right] \frac{d\rho(R)}{dR} \right) R^4 dR$$

(25)

From here the total angular momentum of a rotating configuration will be determined by

$$J_{tot} = I_{tot}(a)\Omega. \tag{26}$$

Thus, we have all the necessary equation to investigate equilibrium configurations of classical white dwarfs.

3 Results and discussion

In equilibrium, a rotating star attains a balance between pressure forces, gravitational forces, and centrifugal forces. In classical physics the magnitude of the centrifugal force is determined by the angular velocity Ω of the fluid relative to a distant observer. In the literature angular velocityΩgiven by (4) is usually known as mass-shedding or Keplerian angular velocity.

In Fig. 1 the mass of a white dwarf is shown as a function of the central density. The mass is given in the units of one solar mass and the central density is given in grams per centimeter cube. We have selected two equations of state: the Chandrasekhar equation of state with average molecular weight $\mu = 2$, and the Salpeter equation of state for carbon and iron white dwarfs as a limiting case. All solid curves indicate non-rotating (static) white dwarfs, whereas all dashed curves indicate rotating white dwarfs at the mass-shedding rate. As it has been expected rotating white dwarfs have larger masses with respect to their static counterparts. In all our computations we restricted the values of the central density to the values of inverse beta decay density to fulfill the stability condition of white dwarfs [14]. In classic white dwarfs the maximum (threshold) value of the central density is considered to be the minimum value between the density for the onset of inverse-beta decay process[13,16].

Figure 2 shows mass and equatorial radius relation. The equatorial radius reduces to the static radius in the non-rotating limit. All legends in the plot are the same as in Fig. 1. Depending on the equation of state and chemical composition, white dwarfs display different mass-radius relation. This explains the variety of observed white dwarfs. Nowadays, we have data for 9316 white dwarfs and all of them have diverse characteristics [17, 18].

Figure 3 illustrates normalized moment of inertia as a function of the central density. The legends are the same as in Fig.1. The lower the density the larger the difference in the moment of inertia of rotating and static white dwarfs.
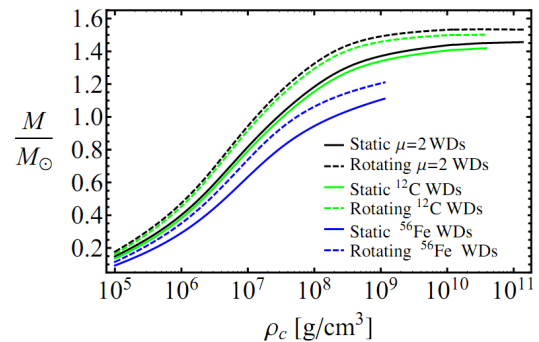

Figure 1 – Mass versus central density
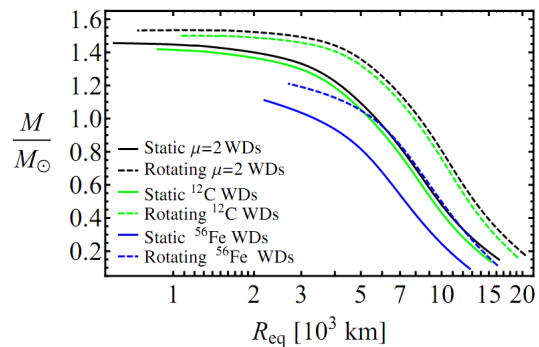(see the color version on the web)

Figure 2 – Mass versus equatorial radius
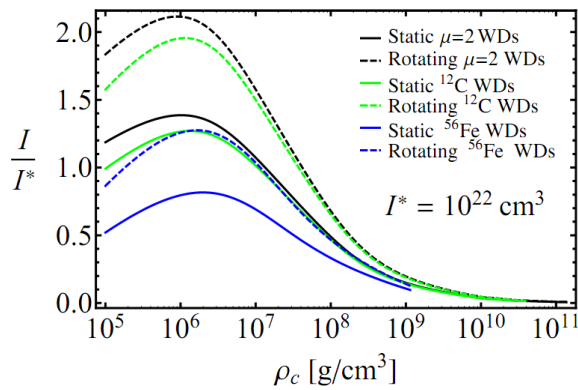(see the color version on the web)

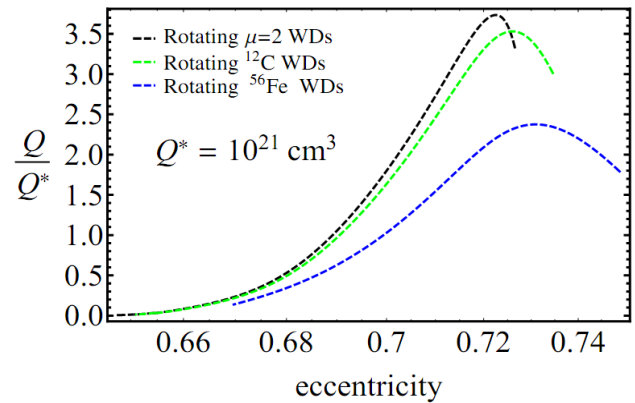Figure 3 – Moment of inertia versus central density
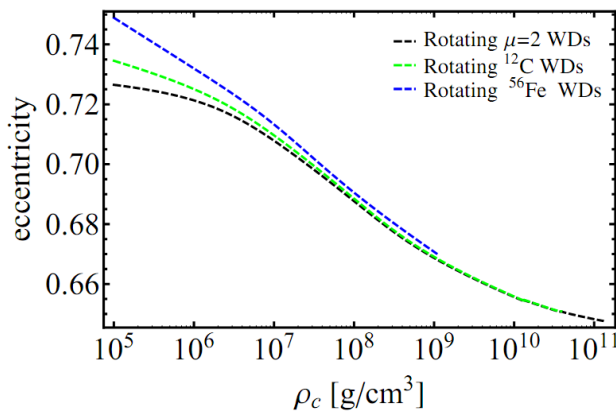

Figure 5 – Quadrupole moment versus eccentricity

Figure 4 – Eccentricity versus central density

4 Conclusion

The equations have been numerically integrated in order to calculate the structure of slowly rotating classical white dwarfs in hydrostatic equilibrium. In particular, the relation between mass and central density, the shapes of rotating stars have been calculated for the Chandrasekhar and Salpeter equations of state.

The equations which determine the moment of inertia and the quadrupole moment of the rotating star have also been integrated numerically. The product of the moment of inertia and the angular velocity determines the angular momentum of a star.

The eccentricity of rotating white dwarfs as a function of central density is shown in Fig. 4. For higher densities the eccentricity decreases and vice versa. Thus, white dwarfs in this case become more spherical as they approach their maximum mass.

The normalized mass quadrupole moment versus eccentricity is represented in Fig. 5. The legends are the same as in Fig. 1. Here one can see that for larger densities the eccentricity and quadrupole moment are correlated. The quadrupole moment decreases as well with increasing density. The system becomes more gravitationally bound.

All these quantities play a fundamental role in describing the equilibrium configurations of uniformly rotating main sequence stars as well as planets. The results obtained in the work are in agreement with other work in the literature related to the rigidly rotating white dwarfs in the Newtonian physics [19-20].

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